Corner critical exponents from the mean-field renormalisation group

1988 J. Phys. A: Math. Gen. 21 L523

(http://iopscience.iop.org/0305-4470/21/9/008)

J. Phys. A: Math. Gen. **21** (1988) L523–L528. Printed in the UK

LETTER TO THE EDITOR

Corner critical exponents from the mean-field renormalisation group

J O Indekeu† and G Menu

Katholieke Universiteit Leuven, Laboratorium voor Molekuulfysika, Celestijnenlaan 200 D, B-3030 Leuven, Belgium

Received 14 March 1988

Abstract. The previously introduced method of mean-field renormalisation has provided a unified approach to bulk and surface critical behaviour. The method is currently used for the computation of the critical exponent associated with the order parameter at corners. Applications to the square and triangular Ising model are presented. The angle dependence of the corner critical exponent is qualitatively reproduced. The results for the bulk critical exponents are more accurate than in previously employed mean-field renormalisation schemes.

The mean-field renormalisation group (MFRG) (Indekeu *et al* 1982, Stella 1984) has been widely used as a tool for semiquantitative computations of bulk critical points and critical exponents. The method is characterised by its easy and broad applicability which originates from its affinity to classical approximations as far as computational input is concerned. The potential power of the approach lies in the combination of classical approximations on the one hand, and scaling and renormalisation group ideas on the other. The areas of application include geometrical critical phenomena and percolation, classical and quantum spin models, both ordered and disordered, dynamical critical phenomena, and surface critical phenomena.

Recently it has been shown that the MFRG can be cast in a form which makes explicit its relationship to standard finite-size scaling. From this development has resulted a unifying approach to *bulk and surface* critical behaviour (Indekeu *et al* 1987). It has turned out that the most natural and consistent implementation of MFRG simultaneously provides estimates for bulk and surface critical exponents. Also the accuracy of the method has been improved because the unifying scheme guarantees convergence to exact results (assuming the validity of standard finite-size scaling).

In the present letter another unifying scheme is developed which allows the computation of the *corner critical exponent*, i.e. the critical exponent which describes the vanishing of the order parameter at a corner when the system approaches criticality in bulk (Cardy 1983). The present approach can be viewed as a refinement of the unifying approach to bulk and surface critical behaviour. Indeed, along the surface of, e.g., a two-dimensional system, we now differentiate between edges and corners, because they give rise to different scaling behaviours for the local-order parameter. Similarly, in a three-dimensional system, one can decompose the surface into faces, edges and corners and distinguish the corresponding scaling properties. From now on attention will be restricted to two-dimensional systems ($d = 2$).

† Research Associate, National Fund for Scientific Research, Belgium.

0305-4470/88/090523+06$02.50

For concreteness, let us consider the ferromagnetic nearest-neighbour Ising model on two-dimensional lattices. The reduced Hamiltonian is

$$-\beta H(s) \equiv H(s) = K \sum_{\langle ij \rangle} s_i s_j + h \sum_i s_i \tag{1}$$

where $s_i = \pm 1$, $\beta = 1/k_B T$, K is the nearest-neighbour coupling, and h is the external magnetic field.

For a cluster of N interacting spins (e.g., a square cluster on the square lattice) the average magnetisation per spin is defined as

$$m_N \equiv \frac{1}{N} \sum_{i=1}^{N} \langle s_i \rangle. \tag{2}$$

Within the MFRG approach m_N is computed in the presence of an *effective magnetisation* b acting on the surface (or boundary) of the cluster. This gives rise to an *effective field* equal to $\alpha K b$ acting on a surface spin with α nearest neighbours outside the cluster. In the example of a square cluster on the square lattice, $\alpha = 1$ for the surface spins along the edges, and $\alpha = 2$ at the corners.

It follows from finite-size-scaling hypotheses that for finite, but large, systems with bulk coupling K, magnetic bulk field h, and with a magnetic *surface field* h_s acting on the surface spins (except at corners), and with a magnetic *corner field* h_c acting on the corner spins, the following homogeneity should hold near criticality for the dimensionless free energy per site f:

$$f_{N'}(K_c + L^{Y_T} \Delta K, L^{Y_H} h, L^{Y_{HS}} h_s, L^{Y_{HC}} h_c) = L^d f_N(K_c + \Delta K, h, h_s, h_c) \tag{3}$$

where K_c denotes the critical value of K. For the two-dimensional Ising model at the 'ordinary' transition the critical exponent Y_{HS} takes the value $\frac{1}{2}$ (Binder 1983) and

$$Y_{HC} = -\pi/2\theta \tag{4}$$

(Barber *et al* 1984) where θ is the angle spanned by the edges of the system. The latter result follows from $\beta_c = \pi/2\theta$ and $\beta_c = (d - 2 - Y_{HC})/Y_T$. Note that the corner exponent depends on the angle and is thus less universal than surface or bulk exponents. We will consider corners with $\theta = \pi/2$, $\pi/3$ and $2\pi/3$. These angles occur most naturally in squares (or rectangles) on the square lattice, and in triangles and diamonds on the triangular lattice.

In order to obtain full consistency between the MFRG method and finite-size scaling, it is important to take into account that in a cluster all surface spins are subjected to an effective field proportional to Kb. In the thermodynamic limit, where the distinction between bulk, edges and corners is well defined, the system is consequently under the influence of a *surface field*

$$h_s \sim Kb \tag{5}$$

and a *corner field*

$$h_c \sim Kb. \tag{6}$$

At this point it is clear that, strictly speaking, we must differentiate between the effective magnetisation b along the edges, b_s, and at corners, b_c, because we must allow different scaling properties for h_s and h_c. Also, it makes sense physically that the effective magnetisations which surround the cluster depend on the local surface geometry. Indeed, similar inhomogeneities are found in a more refined approach where the effective magnetisations do not merely represent the 'mean field', but rather the 'reaction field' which depends on the spin fluctuations close to the surface inside the cluster (Indekeu *et al* 1984).

The distinction between b_s and b_c can be avoided as far as the computation of the surface exponent Y_{HS} is concerned (Indekeu *et al* 1987) because in the thermodynamic limit the corners become negligible compared to the edges. When the distinction between b_s and b_c is made, as is presently done, the unifying bulk-surface approach allows the computation of Y_{HC}.

In view of (5) and (6) we postulate that, close to criticality and for very large clusters, the effective magnetisations must scale like their associated surface fields, i.e.

$$b'_s = L^{Y_{HS}} b_s \tag{7}$$

$$b'_c = L^{Y_{HC}} b_c. \tag{8}$$

As in previous applications of MFRG, it is worth stressing that for *small clusters*, i.e. the clusters one works with in practice, the distinction between bulk and surface (and between edges and corners) is not always easily made. Nevertheless it has often been shown, and it will once more be demonstrated in the present development, that the more consistent the approach is made, the more accurate are the determinations of the bulk critical exponents.

For computing the corner critical exponent the following strategy is proposed. Consider three clusters of sizes N, N' and N'' (in decreasing order) and impose the equations (with $b_s = 0$, $b_c \neq 0$):

$$m_{N'}(K', h', b'_c) = L_1^{d-Y_H} m_N(K, h, b_c) \tag{9a}$$

$$m_{N''}(K'', h'', b''_c) = L_2^{d-Y_H} m_{N'}(K, h, b_c) \tag{9b}$$

$$b'_c = L_1^{Y_{HC}} b_c \tag{10a}$$

$$b''_c = L_2^{Y_{HC}} b_c \tag{10b}$$

where $L_1 \equiv (N/N')^{1/d}$ and $L_2 \equiv (N'/N'')^{1/d}$ according to the traditional definition of length rescaling adopted in MFRG. (This definition is somewhat *ad hoc*, but becomes equivalent to any other sensible definition (e.g., Slotte 1987) as $N, N', N'' \to \infty$ in a regular progression of sizes.)

Note that we now have at our disposal a computational scheme from which K_c, Y_T, Y_H and Y_{HC} can be obtained, in a way which is technically very similar to that employed for obtaining K_c, Y_T, Y_H and Y_{HS} (Indekeu *et al* 1987).

We proceed to apply the method to the square and triangular Ising models. On the square lattice, we work with square clusters ($N = p^2$). Our results are presented in table 1. For comparison, the results obtained previously with the surface-exponent scheme (Indekeu *et al* 1987) are given in table 2.

The next application is devoted to the triangular Ising model. There, different angles can be studied in an elegant manner. First we use equilateral triangles ($N = p(p+1)/2$) and compute the corner exponent for $\theta = \pi/3$. The results are shown in table 3. We have also carried out the surface-exponent scheme and obtained the estimates presented in table 4.

Finally, we make use of diamond cells ($N = p^2$) on the triangular lattice in order to be able to compute the corner exponents for two different angles simultaneously.

Table 1. Results for critical point, bulk and corner critical exponents in the square Ising model using square clusters. The angle at the corners is $\pi/2$.

N	N'	N''	K_c	$Y_T^{N,N'}$	$Y_T^{N',N''}$	$Y_H^{N,N'}$	$Y_H^{N',N''}$	$Y_{HC}^{N,N'}$ $(\pi/2)$	$Y_{HC}^{N',N''}$ $(\pi/2)$
16	9	4	0.469	0.92	0.88	1.78	1.72	−0.57	−0.51
25	16	9	0.459	0.94	0.92	1.80	1.76	−0.63	−0.60
Exact			0.441	1	1	1.875	1.875	−1	−1

Table 2. Results previously reported by Indekeu *et al* (1987) for critical point, bulk and surface critical exponents in the square Ising model using square clusters.

N	N'	N''	K_c	$Y_T^{N,N'}$	$Y_T^{N',N''}$	$Y_H^{N,N'}$	$Y_H^{N',N''}$	$Y_{HS}^{N,N'}$	$Y_{HS}^{N',N''}$
16	9	4	0.425	0.86	0.82	1.68	1.65	0.49	0.52
25	16	9	0.430	0.89	0.87	1.71	1.69	0.49	0.51
Exact			0.441	1	1	1.875	1.875	0.5	0.5

Table 3. Results for critical point, bulk and corner critical exponents, in the triangular Ising model using equilateral trangles. The angle at the corners is $\pi/3$.

N	N'	N''	K_c	$Y_T^{N,N'}$	$Y_T^{N',N''}$	$Y_H^{N,N'}$	$Y_H^{N',N''}$	$Y_{HC}^{N,N'}$ $(\pi/3)$	$Y_{HC}^{N',N''}$ $(\pi/3)$
10	6	3	0.315	0.90	0.85	1.76	1.67	−0.76	−0.67
15	10	6	0.302	0.91	0.88	1.79	1.73	−0.86	−0.80
21	15	10	0.295	0.92	0.90	1.81	1.77	−0.94	−0.90
28	21	15	0.290	0.93	0.91	1.82	1.79	−1.00	−0.97
Exact			0.275	1	1	1.875	1.875	−1.5	−1.5

Table 4. Results for critical point, bulk and surface critical exponents, in the triangular Ising model using equilateral triangles, obtained by the surface-exponent scheme.

N	N'	N''	K_c	$Y_T^{N,N'}$	$Y_T^{N',N''}$	$Y_H^{N,N'}$	$Y_H^{N',N''}$	$Y_{HS}^{N,N'}$	$Y_{HS}^{N',N''}$
10	6	3	0.261	0.80	0.75	1.61	1.55	0.56	0.61
15	10	6	0.265	0.84	0.81	1.66	1.62	0.54	0.57
21	15	10	0.267	0.87	0.85	1.69	1.66	0.52	0.54
28	21	15	0.269	0.89	0.87	1.71	1.69	0.51	0.53
Exact			0.275	1	1	1.875	1.875	0.5	0.5

The diamonds have two corners with $\theta = \pi/3$ and two with $\theta = 2\pi/3$. The results for $\theta = \pi/3$ are displayed in table 5 and those for $\theta = 2\pi/3$ in table 6. Note that the ratio of the corner exponents is the inverse ratio of the angles (4). This ratio is deduced from the data in tables 5 and 6, and the results are shown in table 7. Finally, in table 8 the results are presented which are obtained with the surface-exponent scheme.

We now proceed to discuss our results. The estimates which we have obtained for the corner critical exponent are qualitatively sound but are much less accurate than the estimates for the bulk and surface exponents. The angle dependence of y_{HC} is qualitatively reproduced. The calculation which makes use of the diamond cells allows

Table 5. Results for critical point, bulk and corner critical exponents, in the triangular Ising model using diamond clusters. The corner field is switched on at the two corners with $\pi/3$ angles only.

N	N'	N''	K_c	$Y_T^{N,N'}$	$Y_T^{N',N''}$	$Y_H^{N,N'}$	$Y_H^{N',N''}$	$Y_{HC}^{N,N'}(\pi/3)$	$Y_{HC}^{N',N''}(\pi/3)$
16	9	4	0.305	0.92	0.87	1.81	1.72	−0.85	−0.76
25	16	9	0.295	0.94	0.91	1.83	1.78	−0.95	−0.90
Exact			0.275	1	1	1.875	1.875	−1.5	−1.5

Table 6. Results for critical point, bulk and corner critical exponents, in the triangular Ising model using diamond clusters. The corner field is switched on at the two corners with $2\pi/3$ angles only.

N	N'	N''	K_c	$Y_T^{N,N'}$	$Y_T^{N',N''}$	$Y_H^{N,N'}$	$Y_H^{N',N''}$	$Y_{HC}^{N,N'}(2\pi/3)$	$Y_{HC}^{N',N''}(2\pi/3)$
16	9	4	0.272	0.89	0.84	1.69	1.63	−0.59	−0.53
25	16	9	0.273	0.91	0.89	1.73	1.69	−0.63	−0.59
Exact			0.275	1	1	1.875	1.875	−0.75	−0.75

Table 7. Ratios of corner exponents, in the triangular Ising model using diamond clusters, for two different angles $\pi/3$ and $2\pi/3$.

N	N'	N''	$Y_{HC}^{N,N'}(\pi/3)/Y_{HC}^{N,N'}(2\pi/3)$	$Y_{HC}^{N',N''}(\pi/3)/Y_{HC}^{N',N''}(2\pi/3)$
16	9	4	1.43	1.42
25	16	9	1.52	1.51
Exact			2	2

Table 8. Results for critical point, bulk and surface critical exponents, in the triangular Ising model using diamond clusters, obtained by the surface-exponent scheme.

N	N'	N''	K_c	$Y_T^{N,N'}$	$Y_T^{N',N''}$	$Y_H^{N,N'}$	$Y_H^{N',N''}$	$Y_{HS}^{N,N'}$	$Y_{HS}^{N',N''}$
16	9	4	0.264	0.85	0.80	1.66	1.61	0.52	0.57
25	16	9	0.268	0.88	0.86	1.70	1.68	0.51	0.54
Exact			0.275	1	1	1.875	1.875	0.5	0.5

the direct comparison of $y_{HC}(\pi/3)$ and $y_{HC}(2\pi/3)$, because in that calculation the two angles can be studied simultaneously. The ratios in table 7 exhibit the correct trend, even though they are still far from the exact value.

It is also interesting to compare $y_{HC}(\pi/3)$ obtained using equilateral triangles (table 3) with the same exponent computed using diamonds (table 5). One notes that for comparable pairs of cluster sizes the exponent values are much alike, which is quite satisfactory. It would be interesting to go further along this line of comparison and compute $y_{HC}(2\pi/3)$ also on hexagonal clusters and confront the results with those computed on diamonds. This is not practicable, however, because the hexagons consist of $N''=7$, $N'=19$ and $N=39$ spins, which is too large for our method of exact evaluation of thermal averages.

Another interesting comparison is that between the y_{HC} exponents on the triangular and the square lattice for identical pairs of sizes. When $y_{HC}(\pi/3)$ from table 5 is compared with $y_{HC}(\pi/2)$ from table 1, the ratios turn out to be very nearly exact. This success is, however, fortuitous in view of the disappointing fact that $y_{HC}(\pi/2)$ is hardly distinguishable from $y_{HC}(2\pi/3)$ in table 6.

Another point of interest concerns the value of $\beta_c = -y_{HC}/y_T = \pi/2\theta$ which describes the vanishing of the spontaneous corner magnetisation m_c as the critical point is approached in bulk. For angles $\theta > \pi/2$, $\beta_c < 1$, which means that m_c approaches zero with infinite slope as the critical temperature is approached. This behaviour is similar to that of the bulk or surface magnetisation. For $\theta < \pi/2$ qualitatively different behaviour occurs since $\beta_c > 1$ which signifies an approach with zero slope. Note that, in spite of our generally poor numerical values of y_{HC}, the two qualitatively different behaviours are emerging. Indeed, the results for large sizes in tables 3 and 5 already clearly show that $\beta_c > 1$ for $\theta = \pi/3$.

Finally, a remarkable feature apparent from the tables and from comparisons with earlier MFRG works is that the bulk exponents are obtained more accurately with the present corner-exponent scheme than with previously developed schemes.

It is a great pleasure to thank Professor P W Kasteleyn for valuable remarks which lie at the basis of the present refinement of the MFRG methodology.

References

Barber M N, Peschel I and Pearce P A 1984 *J. Stat. Phys.* **37** 497
Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic) p 1
Cardy J H 1983 *J. Phys. A: Math. Gen.* **16** 3617
Indekeu J O, Maritan A and Stella A L 1982 *J. Phys. A: Math. Gen.* **15** L291
—— 1987 *Phys. Rev.* B **35** 305
Indekeu J O, Stella A L and Zhang L 1984 *J. Phys. A: Math. Gen.* **17** L341
Slotte P A 1987 *J. Phys. A: Math. Gen.* **20** L117
Stella A L 1984 *Group Theoretical Methods in Physics* (*Lecture Notes in Physics* **210**) ed G Denardo, G Ghirardi and T Weber (Berlin: Springer) p 448